

Long time estimate of solutions to 3d Navier-Stokes equations coupled with the heat convection

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Abstract. We examine the Navier-Stokes equations with homogeneous slip boundary conditions coupled with the heat equation with homogeneous Neumann conditions in a bounded domain in \mathbb{R}^3 . The considered domain is a cylinder with x_3 -axis. The aim of this paper is to show long time estimates without smallness of the initial velocity, the initial temperature and the external force. To prove the estimate we need however smallness of L_2 norms of derivatives with respect to x_3 of the initial velocity, the initial temperature and the external force.

Key words: Navier-Stokes equations, heat equation, coupled, slip boundary conditions, the Neumann condition, long time estimate, regular solutions

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1. Introduction

The aim of this paper is to derive long time a priori estimate for some initial-boundary value problem for a system of the Navier-Stokes equations coupled with the heat equation. We assume the slip boundary conditions for the Navier-Stokes equations and the Neumann condition for the heat equations. We examine the problem in a straight finite cylinder. To obtain the estimate we follow the ideas from [5, 7, 8] and the considered solution remains close to a two-dimensional solution. The estimate is the first and the most important step to prove the existence of solutions to the problem (see (1.1)) by the Leray-Schauder fixed point theorem (see the next paper of the authors [6]).

We consider the following problem

$$\begin{aligned}
 (1.1) \quad & v_{,t} + v \cdot \nabla v - \operatorname{div} \mathbf{T}(v, p) = \alpha(\theta)f \quad \text{in } \Omega^T = \Omega \times (0, T), \\
 & \operatorname{div} v = 0 \quad \text{in } \Omega^T, \\
 & \theta_{,t} + v \cdot \nabla \theta - \varkappa \Delta \theta = 0 \quad \text{in } \Omega^T, \\
 & \bar{n} \cdot \mathbf{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 \quad \text{on } S^T = S \times (0, T), \\
 & \bar{n} \cdot \bar{v} = 0 \quad \text{on } S^T, \\
 & \bar{n} \cdot \nabla \theta = 0 \quad \text{on } S^T, \\
 & v|_{t=0} = v(0), \quad \theta|_{t=0} = \theta(0) \quad \text{in } \Omega,
 \end{aligned}$$

where by $x = (x_1, x_2, x_3)$ we denote the Cartesian coordinates, $\Omega \subset \mathbb{R}^3$ is a cylindrical type domain parallel to the axis x_3 with arbitrary cross section, $S = \partial\Omega$, $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid motion, $p = p(x, t) \in \mathbb{R}^1$ the pressure, $\theta = \theta(x, t) \in \mathbb{R}_+$ the temperature, $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ the external force field, \bar{n} is the unit outward normal vector to the boundary S , $\bar{\tau}_\alpha$, $\alpha = 1, 2$, are tangent vectors to S and the dot denotes the scalar product in \mathbb{R}^3 . We define the stress tensor by

$$\mathbf{T}(v, p) = \nu \mathbf{D}(v) - p \mathbf{I},$$

where ν is the constant viscosity coefficient, \mathbf{I} is the unit matrix and $\mathbf{D}(v)$ is the dilatation tensor of the form

$$\mathbf{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally \varkappa is a positive heat conductivity coefficient.

We assume that $S = S_1 \cup S_2$, where S_1 is the part of the boundary which is parallel to the axis x_3 and S_2 is perpendicular to x_3 . Hence

$$S_1 = \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) = c_*, -b < x_3 < b\}$$

and

$$S_2 = \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_*, x_3 \text{ is equal either to } -b \text{ or } b\},$$

where b, c_* are positive given numbers and $\varphi_0(x_1, x_2)$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$. We can assume $\bar{\tau}_1 = (\tau_{11}, \tau_{12}, 0)$, $\bar{\tau}_2 = (0, 0, 1)$ and $\bar{n} = (\tau_{12}, -\tau_{11}, 0)$ on S_1 . Assume that $\alpha \in C^2(\mathbb{R})$ and Ω^T satisfies the weak l -horn condition, where $l = (2, 2, 1)$ (see [2, Ch. 2, Sect. 8]).

Moreover we assume that Ω^T is not axially symmetric. Now we formulate the main result of this paper. Let $g = f_{,x_3}$, $h = v_{,x_3}$, $q = p_{,x_3}$, $\vartheta = \theta_{,x_3}$, $\chi = (\text{rot } v)_3$, $f = (\text{rot } f)_3$. Assume that $\|\theta(0)\|_{L_\infty(\Omega)} < \infty$. Define

$$a : [0, \infty) \rightarrow [0, \infty), \quad a(x) = \sup\{|\alpha(y)| + |\alpha'(y)| : |y| \leq x\}$$

and

$$(1.1') \quad a(\theta(x)) \leq c_1,$$

where $c_1 = a(\|\theta(0)\|_{L_\infty(\Omega)})$. The inequality (1.1') is justified in view of Lemma 2.3, Remark 2.4 and properties of function $a(x)$. Moreover assume that $\frac{5}{3} < \sigma < \infty$, $\frac{5}{3} < \varrho < \infty$, $\frac{5}{\varrho} - \frac{5}{\sigma} < 1$ and for $t \leq T$

1. $c_1 \|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1 c_0 \|f\|_{L_\infty(0,t;L_3(\Omega))} + c_1 \|F\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1 \|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)} + \psi(c_0) + c_0^2(c_1 \|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \leq k_1 < \infty,$
2. $\|f\|_{L_2(0,t;L_3(\Omega))} \leq k_2 < \infty,$
3. $\|f\|_{L_2(\Omega^t)} + \|v(0)\|_{H^1(\Omega)} \leq k_3 < \infty,$
4. $c_1 \|f\|_{L_\infty(\Omega^t)} e^{cc_1^2 k_2^2} k_1 + c_1 \|g\|_{L_\sigma(\Omega^t)} + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} \leq k_4 < \infty,$
5. $c_1 \|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1 \|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} \leq d < \infty$
6. $c_1 + \|f\|_{L_\varrho(\Omega^t)} + \|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} \leq k_5 < \infty,$
where c_0 is a constant from Lemma 2.2, and ψ_0 is an increasing function from Lemma 3.3 and k_1, \dots, k_5 are given constants.

Main Theorem. For every fixed T , conditions 1–6 with constants $k_1 - k_5$, c_0, c_1 there exist a sufficiently small d and a constant $B = B(k_1, \dots, k_5, c_0, c_1) < \infty$ such that for any strong solution v, p, θ to problem (1.1) we have

$$(1.2) \quad \|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\nabla p\|_{L_\varrho(\Omega^t)} + \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \leq B,$$

$$(1.3) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} + \|\vartheta\|_{W_\sigma^{2,1}(\Omega^t)} \leq B.$$

$t \leq T$.

The result is necessary to prove a long time existence of regular solutions to (1.1) in [6].

2. Preliminaries

In this section we introduce notation and basic estimates for weak solutions to problem (1.1).

2.1. Notation

We use isotropic and anisotropic Lebesgue spaces: $L_p(Q)$, $Q \in \{\Omega^T, S^T, \Omega, S\}$, $p \in [1, \infty]$; $L_q(0, T; L_p(Q))$, $Q \in \{\Omega, S\}$, $p, q \in [1, \infty]$; Sobolev spaces

$$W_q^{s, s/2}(Q^T), \quad Q \in \{\Omega, S\}, \quad q \in [1, \infty], \quad s \in \mathbb{N} \cup \{0\}$$

with the norm

$$\|u\|_{W_q^{s, s/2}(Q^T)} = \left(\sum_{|\alpha|+2a \leq s} \int |D_x^\alpha \partial_t^a u|^q dx dt \right)^{1/q},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $a, \alpha_i \in \mathbb{N} \cup \{0\}$.

In the case $q = 2$

$$H^s(Q) = W_2^s(Q), \quad H^{s, s/2}(Q^T) = W_2^{s, s/2}(Q^T), \quad Q \in \{\Omega, S\}.$$

Moreover, $L_2(Q) = H^0(Q)$, $L_p(Q) = W_p^0(Q)$, $L_p(Q^T) = W_p^{0,0}(Q^T)$.

We define a space natural for study weak solutions to the Navier-Stokes and parabolic equations

$$V_2^k(\Omega^T) = \left\{ u : \|u\|_{V_2^k(\Omega^T)} = \text{esssup}_{t \in [0, T]} \|u\|_{H^k(\Omega)} + \left(\int_0^T \|\nabla u\|_{H^k(\Omega)}^2 dt \right)^{1/2} < \infty \right\}.$$

2.2. Weak solutions

By a weak solution to problem (1.1) we mean $v \in V_2^0(\Omega^T)$, $\theta \in V_2^0(\Omega^T) \cap L_\infty(\Omega^T)$ satisfying the integral identities

$$\begin{aligned} (2.1) \quad & - \int_{\Omega^T} v \cdot \varphi_{,t} dx dt + \int_{\Omega^T} v \cdot \nabla v \cdot \varphi dx dt + \frac{\nu}{2} \int_{\Omega^T} \mathbf{D}(v) \cdot \mathbf{D}(\varphi) dx dt \\ & = \int_{\Omega^T} \alpha(\theta) f \cdot \varphi dx dt + \int_{\Omega} v(0) \varphi(0) dx, \end{aligned}$$

$$\begin{aligned}
(2.2) \quad & - \int_{\Omega^T} \theta \psi_{,t} dx dt + \int_{\Omega^T} v \cdot \nabla \theta \psi dx dt + \varkappa \int_{\Omega^T} \nabla \theta \cdot \nabla \psi dx dt \\
& = \int_{\Omega} \theta(0) \psi(0) dx,
\end{aligned}$$

which hold for $\varphi, \psi \in W_2^{1,1}(\Omega^T) \cap L_5(\Omega^T)$ such that $\varphi(T) = 0$, $\psi(T) = 0$, $\operatorname{div} \varphi = 0$, $\varphi \cdot \bar{n}|_S = 0$.

Lemma 2.1. (the Korn inequality, see [10]) Assume that

$$(2.3) \quad E_{\Omega}(v) = \|\mathbf{D}(v)\|_{L_2(\Omega)}^2 < \infty, \quad v \cdot \bar{n}|_S = 0, \quad \operatorname{div} v = 0.$$

If Ω is not axially symmetric there exists a constant c_1 such that

$$(2.4) \quad \|v\|_{H^1(\Omega)}^2 \leq c_1 E_{\Omega}(v).$$

If Ω is axially symmetric, $\eta = (-x_2, x_1, 0)$, $\alpha = \int_{\Omega} v \cdot \eta dx$, then there exists a constant c_2 such that

$$(2.5) \quad \|v\|_{H^1(\Omega)}^2 \leq c_2 (E_{\Omega}(v) + |\alpha|^2).$$

Let us consider the problem

$$\begin{aligned}
(2.5') \quad & h_{,t} - \operatorname{div} \mathbf{T}(h, q) = f && \text{in } \Omega^T, \\
& \operatorname{div} h = 0 && \text{in } \Omega^T, \\
& \bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbf{D}(h) \cdot \bar{\tau}_{\alpha} = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\
& h_i = 0, \quad i = 1, 2, \quad h_{3,x_3} = 0 && \text{on } S_2^T, \\
& h|_{t=0} = h(0) && \text{in } \Omega
\end{aligned}$$

Lemma 2.2. Let $f \in L_p(\Omega^T)$, $h(0) \in W_p^{2-2/p}(\Omega)$, $S \in C^2$, $q < p < \infty$. Then there exists a solution to problem (2.5') such that $h \in W_p^{2,1}(\Omega^T)$, $\nabla q \in L_p(\Omega^T)$ and there exists a constant c depending on S and p such that

$$\begin{aligned}
(2.6') \quad & \|h\|_{W_p^{2,1}(\Omega^T)} + \|\nabla q\|_{L_p(\Omega^T)} \leq c(\|f\|_{L_p(\Omega^T)} \\
& + \|h(0)\|_{W_p^{2-2/p}(\Omega)}).
\end{aligned}$$

The proof is similar to the proof from [1].

Lemma 2.3. Assume that $v(0) \in L_2(\Omega)$, $\theta(0) \in L_{\infty}(\Omega)$, $f \in L_2(0, T; L_{6/5}(\Omega))$, $T < \infty$. Assume that Ω is not axially symmetric. Assume that there exist constants θ_*, θ^* such that $\theta_* < \theta^*$ and $\theta_* \leq \theta_0(x) \leq \theta^*$, $x \in \Omega$.

Then there exists a weak solution to problem (1.1) such that $(v, \theta) \in V_2^0(\Omega^T) \times V_2^0(\Omega^T)$, $\theta \in L_\infty(\Omega^T)$ and

$$(2.6) \quad \theta_* \leq \theta(x, t) \leq \theta^*, \quad (x, t) \in \Omega^T,$$

$$(2.7) \quad \|v\|_{V_2^0(\Omega^T)} \leq c(a(\|\theta_0\|_{L_\infty(\Omega)})\|f\|_{L_2(0,T;L_{6/5}(\Omega))} + \|v_0\|_{L_2(\Omega)}) \leq c_0,$$

$$(2.8) \quad \|\theta\|_{V_2^0(\Omega^T)} \leq c\|\theta_0\|_{L_2(\Omega)} \leq c_0.$$

Proof. Estimate (2.6) follows from standard considerations (see [5, Lemmas 3.1, 3.2]). Estimates (2.7), (2.8) follow formally from (1.1)_{1,3} by multiplying them by v and θ , respectively, integrating over Ω and $(0, t)$, $t \in (0, T)$, employing (2.6), (1.1)₂ and using the boundary and initial conditions (1.1)_{4,5,6,7}.

Existence can be shown in the same way as in [3, Ch. 3, Sect. 1–5].

This concludes the proof. \square

Remark 2.4. If $\theta(0) \geq 0$, then $\theta(t) \geq 0$ for $t \geq 0$.

2.3. Auxiliary problems

To prove the existence of global regular solutions we distinguish the quantities

$$(2.9) \quad h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3}, \quad \vartheta = \theta_{,x_3}.$$

Differentiating (1.1)_{1,2,4,5} with respect to x_3 and using [1, 2] yields

$$(2.10) \quad \begin{aligned} h_{,t} - \operatorname{div} \mathbf{T}(h, q) &= -v \cdot \nabla h - h \cdot \nabla v + \alpha_\theta \vartheta f + \alpha g && \text{in } \Omega^T, \\ \operatorname{div} h &= 0 && \text{in } \Omega^T, \\ \bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbf{D}(h) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\ h_i = 0, \quad i = 1, 2, \quad h_{3,x_3} &= 0 && \text{on } S_2^T, \\ h|_{t=0} &= h(0) && \text{in } \Omega. \end{aligned}$$

Let q and f_3 be given, Then $w = v_3$ is a solution to the problem

$$(2.11) \quad \begin{aligned} w_{,t} + v \cdot \nabla w - \nu \Delta w &= -q + \alpha(\theta) f_3 && \text{in } \Omega^T, \\ \bar{n} \cdot \nabla w &= 0 && \text{on } S_1^T, \\ w &= 0 && \text{on } S_2^T, \\ w|_{t=0} &= w(0) && \text{in } \Omega. \end{aligned}$$

Let $F = (\operatorname{rot} f)_3$, h, v, w be given. Then $\chi = (\operatorname{rot} v)_3$ is a solution to the problem (see [8])

$$\begin{aligned}
(2.12) \quad & \chi_{,t} + v \cdot \nabla \chi - h_3 \chi + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu \Delta \chi \\
& = \alpha_\theta (\theta_{,x_1} f_2 - \theta_{,x_2} f_1) + \alpha F && \text{in } \Omega^T, \\
& \chi = v_i (n_{i,x_j} \tau_{1j} + \tau_{1i,x_j} n_j) + v \cdot \bar{\tau}_1 (\tau_{12,x_1} - \tau_{11,x_2}) \\
& \equiv \chi_* && \text{on } S_1^T, \\
& \chi_{,x_3} = 0 && \text{on } S_2^T, \\
& \chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}$$

where the summation convention over repeated indices is assumed. Differentiating (1.1)_{3,6,7} with respect to x_3 yields

$$\begin{aligned}
(2.13) \quad & \vartheta_{,t} + v \cdot \nabla \vartheta + h \cdot \nabla \theta - \varkappa \Delta \vartheta = 0 && \text{in } \Omega^T, \\
& \bar{n} \cdot \nabla \vartheta = 0 && \text{on } S_1^T, \\
& \vartheta = 0 && \text{on } S_2^T, \\
& \vartheta|_{t=0} = \vartheta(0) && \text{in } \Omega.
\end{aligned}$$

Lemma 2.5. Assume that $\mathbf{D}(h) \in L_2(\Omega)$, $h \cdot \bar{n}|_S = 0$, $\operatorname{div} h = 0$. Then h satisfies the inequality

$$(2.14) \quad \|h\|_{H^1(\Omega)} \leq c \|\mathbf{D}(h)\|_{L_2(\Omega)}.$$

where c is a constant independent of h .

Proof. To show (2.14) we examine the expression

$$\int_{\Omega} |\mathbf{D}(h)|^2 dx = \int_{\Omega} (h_{i,x_j} + h_{j,x_i})^2 dx = \int_{\Omega} (2h_{i,x_j}^2 + 2h_{i,x_j} h_{j,x_i}) dx,$$

where the second expression implies

$$\begin{aligned}
\int_{\Omega} h_{i,x_j} h_{j,x_i} dx &= \int_{\Omega} (h_{i,x_j} h_j)_{,x_i} dx - \int_{\Omega} h_{i,x_i x_j} h_j dx = \int_{S_1 \cup S_2} n_i h_{i,x_j} h_j dS \\
&= - \int_{S_1} n_{i,x_j} h_i h_j dS_1 + \int_{S_2} n_i h_{i,x_j} h_j dS_2 = - \int_{S_1} n_{i,x_j} h_i h_j dS_1.
\end{aligned}$$

From the above considerations we have

$$(2.15) \quad \|\nabla h\|_{L_2(\Omega)}^2 \leq c \int_{\Omega} |\mathbf{D}(h)|^2 dx + c \|h\|_{L_2(S_1)}^2.$$

By the trace theorem

$$(2.16) \quad \|\nabla h\|_{L_2(\Omega)}^2 \leq c(\|\mathbf{D}(h)\|_{L_2(\Omega)}^2 + \|h\|_{L_2(\Omega)}^2).$$

From [9] we have that

$$(2.17) \quad \|h\|_{L_2(\Omega)} \leq \delta \|\nabla h\|_{L_2(\Omega)} + M \|\mathbf{D}(h)\|_{L_2(\Omega)},$$

where δ can be chosen sufficiently small and $M = M(\delta)$ is some constant. From (2.15)-(2.17) we have

$$(2.18) \quad \|\nabla h\|_{L_2(\Omega)}^2 \leq c \|\mathbf{D}(h)\|_{L_2(\Omega)}^2.$$

From (2.18) and (2.17) we obtain (2.14). This concludes the proof. \square

Let us consider the elliptic problem

$$(2.22) \quad \begin{aligned} v_{2,x_1} - v_{1,x_2} &= \chi & \text{in } \Omega \subset \mathbb{R}^2, \\ v_{1,x_1} + v_{2,x_2} &= -h_3 & \text{in } \Omega \subset \mathbb{R}^2, \\ v \cdot \bar{n} &= 0 & \text{on } \partial\Omega. \end{aligned}$$

where x_3 is treated as a parameter.

Lemma 2.6. *Assume that $\chi, h_3 \in L_2(\Omega)$. Then $v \in H^1(\Omega)$ and*

$$(2.23) \quad \|v\|_{H^1(\Omega)} \leq c(\|\chi\|_{L_2(\Omega)} + \|h_3\|_{L_2(\Omega)}).$$

Assume that $\chi, h_3 \in H^1(\Omega)$. Then $v \in H^2(\Omega)$ and

$$(2.24) \quad \|v\|_{H^2(\Omega)} \leq c(\|\chi\|_{H^1(\Omega)} + \|h_3\|_{H^1(\Omega)}).$$

Proof. To solve problem (2.22) we introduce the potential φ, ψ such that

$$(2.25) \quad \begin{aligned} v_1 &= \varphi_{,x_1} + \psi_{,x_2}, \\ v_2 &= \varphi_{,x_2} - \psi_{,x_1}. \end{aligned}$$

Using representation (2.25) we see that (2.22)₃ takes the form

$$(2.26) \quad \bar{n} \cdot \nabla \varphi + \bar{\tau} \cdot \nabla \psi = 0 \quad \text{on } S,$$

where $\bar{n} \perp TS$, $\bar{\tau} \in TS$.

The potential φ and ψ are determined up to an arbitrary constant. Moreover, to determine the potential we split boundary condition (3.26) into two boundary conditions

$$(2.27) \quad \begin{aligned} \bar{n} \cdot \nabla \varphi|_S &= 0 \\ \bar{\tau} \cdot \nabla \psi|_S &= 0 \Rightarrow \psi|_S = 0. \end{aligned}$$

Having $v = (v_1, v_2)$ given we calculate φ and ψ from the problems

$$(2.28) \quad \begin{aligned} \Delta\varphi &= v_{1,x_1} + v_{2,x_2} \quad \text{in } \Omega, \\ \bar{n} \cdot \nabla\varphi|_S &= 0, \\ \int_{\Omega} \varphi dx &= 0 \end{aligned}$$

and

$$(2.29) \quad \begin{aligned} \Delta\psi &= v_{1,x_2} - v_{2,x_1} \\ \psi|_S &= 0. \end{aligned}$$

In view of (2.28), (2.29) problem (2.22) takes the form

$$(2.30) \quad \begin{aligned} \Delta\psi &= \chi \quad \psi|_S = 0, \\ \Delta\varphi &= -h_3 \quad \bar{n} \cdot \nabla\varphi|_S = 0, \quad \int_{\Omega} \varphi dx = 0 \end{aligned}$$

Solving problems (2.30) we have the estimates

$$(2.31) \quad \begin{aligned} \|\psi\|_{H^2(\Omega)} &\leq c\|\chi\|_{L_2(\Omega)}, \\ \|\varphi\|_{H^2(\Omega)} &\leq c\|h_3\|_{L_2(\Omega)}. \end{aligned}$$

Hence in view of (2.25) we get (2.23).

For more regular χ and h_3 we have also the estimates

$$(2.32) \quad \begin{aligned} \|\psi\|_{H^3(\Omega)} &\leq c\|\chi\|_{H^1(\Omega)}, \\ \|\varphi\|_{H^3(\Omega)} &\leq c\|h_3\|_{H^1(\Omega)}. \end{aligned}$$

Then (2.25) implies (2.24). This concludes the proof. \square

Now we formulate the result on local existence of solutions to problem (1.1) with regularity allowed by the regularity of data formulated in the Main Theorem. The aim of the result is such that derived in this paper estimates are not only a priori type estimates.

Lemma 2.7. *Let the assumptions of the Main Theorem hold.*

Then for any $A > 0$ there exists $t_ > 0$ and (v, θ, p) -solution to problem (1.1) such that $v \in W_{\varrho}^{2,1}(\Omega^{t_*})$, $\theta \in W_{\varrho}^{2,1}(\Omega^{t_*})$, $\nabla p \in L_{\varrho}(\Omega^{t_*})$, $h \in W_{\sigma}^{2,1}(\Omega^{t_*})$, $\nabla q \in L_{\sigma}(\Omega^{t_*})$ and*

$$\begin{aligned} \|h\|_{W_{\sigma}^{2,1}(\Omega^{t_*})} + \|\nabla q\|_{L_{\sigma}(\Omega^{t_*})} &\leq A, \\ \|v\|_{W_{\varrho}^{2,1}(\Omega^{t_*})} + \|\theta\|_{W_{\varrho}^{2,1}(\Omega^{t_*})} + \|\nabla p\|_{L_{\varrho}(\Omega^{t_*})} &\leq A. \end{aligned}$$

Consider the problem

$$u_{,t} - \nu \Delta u = 0$$

$$u|_S = \varphi$$

$$u|_{t=0} = 0$$

Lemma 2.8. (see [4]) Let $\varphi \in L_q(0, T; L_p(S))$, $p, q \in [1, \infty]$ then $u \in L_q(0, T; L_p(\Omega))$ and

$$\|u\|_{L_q(0, T; L_p(\Omega))} \leq c \|\varphi\|_{L_q(0, T; L_p(S))}.$$

Let $\varphi \in W_2^{\frac{1}{2}, \frac{1}{4}}(S^T)$ then $u \in W_2^{1, 1/2}(\Omega^T)$ and

$$\|u\|_{W_2^{1, 1/2}(\Omega^T)} \leq c \|\varphi\|_{W_2^{1/2, 1/4}(S^T)}.$$

3. Estimates

Lemma 3.1. Let the assumptions of Lemma 2.3 be satisfied. Moreover assume that $f \in L_2(0, T; L_3(\Omega))$, $f_3 \in L_3(0, T; L_{\frac{4}{3}}(S_2))$, $g \in L_2(0, T; L_{6/5}(\Omega))$, $h(0) \in L_2(\Omega)$, $\vartheta(0) \in L_2(\Omega)$, $\nabla v \in L_2(0, T; L_3(\Omega))$, $\nabla \theta \in L_2(0, T; L_3(\Omega))$. Assume that solutions to (2.10), (2.13) are sufficiently regular. Let $c_1 = a(\|\theta_0\|_{L_\infty})$.

Let $h \in (0, T; L_3(\Omega))$, then solutions of (2.10), (2.13) satisfy the inequality

$$(3.1) \quad \begin{aligned} & \|h\|_{V_2^0(\Omega^T)}^2 + \|\vartheta\|_{V_2^0(\Omega^T)}^2 \leq c \exp(cc_1^2 \|f\|_{L_2(0, t; L_3(\Omega))}^2) \\ & \cdot [c_0^2 \|h\|_{L_\infty(0, t; L_3(\Omega))}^2 + c_1^2 \|g\|_{L_2(0, t; L_{6/5}(\Omega))}^2 + c_1^2 \|f_3\|_{L_2(0, t; L_{4/3}(S_2))}^2 \\ & + \|h(0)\|_{L_2(\Omega)}^2 + \|\vartheta(0)\|_{L_2(\Omega)}^2], \quad t \leq T. \end{aligned}$$

Let, additionally, $v, \theta \in L_2(0, T; W_3^1(\Omega))$, then for solutions of (2.10), (2.13) we have

$$(3.2) \quad \begin{aligned} & \|h\|_{V_2^0(\Omega^T)}^2 + \|\vartheta\|_{V_2^0(\Omega^T)}^2 \leq c \exp[c(\|\nabla v\|_{L_2(0, t; L_3(\Omega))}^2 \\ & + \|\nabla \theta\|_{L_2(0, t; L_3(\Omega))}^2 + c_1^2 \|f\|_{L_2(0, t; L_3(\Omega))}^2)] \\ & \cdot [c_1^2 \|g\|_{L_2(0, t; L_{6/5}(\Omega))}^2 + c_1^2 \|f_3\|_{L_2(0, t; L_{4/3}(S_2))}^2 + \|h(0)\|_{L_2(\Omega)}^2 + \|\vartheta(0)\|_{L_2(\Omega)}^2] \end{aligned}$$

$t \leq T$.

Proof. Multiplying (2.10) by h , integrating over Ω and using Lemma 2.5 yields

$$(3.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|h\|_{L_2(\Omega)}^2 + \nu \|h\|_{H^1(\Omega)}^2 \leq c \int_{\Omega} |h \cdot \nabla v \cdot h| dx + c \int_{\Omega} |\alpha_\theta \vartheta f h| dx \\ & + c \int_{\Omega} |\alpha g h| dx + c \int_{S_2} |\alpha f_3 h_3| dx_1 dx_2 \end{aligned}$$

where the first term on the r.h.s. we estimate by

$$\varepsilon_1 \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon_1) \|\nabla v\|_{L_2(\Omega)}^2 \|h\|_{L_3(\Omega)}^2,$$

the second by

$$\varepsilon_2 \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon_2) a^2 (\|\theta_0\|_{L_\infty(\Omega)}) \|\vartheta f\|_{L_{6/5}(\Omega)}^2,$$

the third by

$$\varepsilon_3 \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon_3) a^2 (\|\theta_0\|_{L_\infty(\Omega)}) \|g\|_{L_{6/5}(\Omega)}^2$$

and the fourth by

$$\varepsilon_4 \|h\|_{H^1(\Omega)}^2 + c(1/\varepsilon_4) a^2 (\|\theta_0\|_{L_\infty(\Omega)}) \|f_3\|_{L_{\frac{4}{3}}(S_2)}^2.$$

Assuming that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are sufficiently small we obtain

$$(3.4) \quad \begin{aligned} \frac{d}{dt} \|h\|_{L_2(\Omega)}^2 + \nu \|h\|_{H^1(\Omega)}^2 &\leq c(\|\nabla v\|_{L_2(\Omega)}^2 \|h\|_{L_3(\Omega)}^2 \\ &+ c_1^2 (\|\vartheta\|_{L_2(\Omega)}^2 \|f\|_{L_3(\Omega)}^2 + \|g\|_{L_{6/5}(\Omega)}^2 + \|f_3\|_{L_{\frac{4}{3}}(S_2)}^2)). \end{aligned}$$

Multiplying (2.13) by ϑ and integrating over Ω yields

$$(3.5) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\vartheta\|_{L_2(\Omega)}^2 + \varkappa \|\vartheta\|_{H^1(\Omega)}^2 &\leq c \int_{\Omega} |h \cdot \nabla \theta \vartheta| dx \\ &\leq \varepsilon \|\vartheta\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \|h\|_{L_3(\Omega)}^2 \|\nabla \theta\|_{L_2(\Omega)}^2. \end{aligned}$$

For sufficiently small ε we have

$$(3.6) \quad \frac{d}{dt} \|\vartheta\|_{L_2(\Omega)}^2 + \varkappa \|\vartheta\|_{H^1(\Omega)}^2 \leq c \|h\|_{L_3(\Omega)}^2 \|\nabla \theta\|_{L_2(\Omega)}^2.$$

Adding (3.4) and (3.6), integrating with respect to time and using (2.7) and (2.8) we obtain (3.1).

We can replace inequalities (3.4) and (3.6) by

$$(3.7) \quad \begin{aligned} \frac{d}{dt} \|h\|_{L_2(\Omega)}^2 + \nu \|h\|_{H^1(\Omega)}^2 &\leq c(\|\nabla v\|_{L_3(\Omega)}^2 \|h\|_{L_2(\Omega)}^2 \\ &+ c_1^2 (\|\vartheta\|_{L_2(\Omega)}^2 \|f\|_{L_3(\Omega)}^2 + \|g\|_{L_{6/5}(\Omega)}^2 + \|f_3\|_{L_{\frac{4}{3}}(S_2)}^2)) \end{aligned}$$

and

$$(3.8) \quad \frac{d}{dt} \|\vartheta\|_{L_2(\Omega)}^2 + \varkappa \|\vartheta\|_{H^1(\Omega)}^2 \leq c \|\nabla \theta\|_{L_3(\Omega)}^2 \|h\|_{L_2(\Omega)}^2.$$

Adding (3.7) and (3.8), integrating the sum with respect to time yields (3.2). This ends the proof. \square

To obtain an estimate for solutions to problem (2.12) we introduce a function $\tilde{\chi} : \Omega \times [0, T] \rightarrow \mathbb{R}$ as a solution to the problem

$$(3.9) \quad \begin{aligned} \tilde{\chi}_{,t} - \nu \Delta \tilde{\chi} &= 0 & \text{in } \Omega^T, \\ \tilde{\chi} &= \chi_* & \text{on } S_1^T, \\ \tilde{\chi}_{,x_3} &= 0 & \text{on } S_2^T, \\ \tilde{\chi}|_{t=0} &= 0 & \text{in } \Omega. \end{aligned}$$

Then the function

$$(3.10) \quad \chi' = \chi - \tilde{\chi}$$

satisfies

$$(3.11) \quad \begin{aligned} \chi'_{,t} + v \cdot \nabla \chi' - h_3 \chi' + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu \Delta \chi' \\ = \alpha_\theta (\theta_{,x_1} f_2 - \theta_{,x_2} f_1) + \alpha F - v \cdot \nabla \tilde{\chi} + h_3 \tilde{\chi} & \quad \text{in } \Omega^T, \\ \chi' &= 0 & \text{on } S_1^T, \\ \chi'_{,x_3} &= 0 & \text{on } S_2^T, \\ \chi'|_{t=0} &= \chi(0) & \text{in } \Omega. \end{aligned}$$

Lemma 3.2. *Let the assumptions of Lemma 2.3 be satisfied. Moreover assume that $h, f \in L_\infty(0, T; L_3(\Omega))$, $F \in L_2(0, T; L_{6/5}(\Omega))$, $v' = (v_1, v_2) \in L_\infty(0, T; H^{1/2+\varepsilon}(\Omega)) \cap W_2^{1,1/2}(\Omega^T)$, $\chi(0) \in L_2(\Omega)$, $\varepsilon > 0$ is arbitrary small.*

Assume that solutions to (1.1) are sufficiently regular. Then for solutions to (2.12) we have

$$(3.12) \quad \begin{aligned} \|\chi\|_{V_2^0(\Omega^t)}^2 &\leq c \left(c_0^2 \sup_t \|h\|_{L_3(\Omega)}^2 + c_1^2 c_0^2 \sup_t \|f\|_{L_3(\Omega)}^2 \right. \\ &\quad + c_1^2 \|F\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + c_0^2 \varepsilon_7^2 \|v'\|_{L_\infty(0,t;H^1(\Omega))}^2 \\ &\quad + \|v'\|_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))}^2 + \|v'\|_{W_2^{1,1/2}(\Omega^t)}^2 + \|\chi(0)\|_{L_2(\Omega)}^2 \\ &\quad \left. + \left(c_0^2 c^2 \left(\frac{1}{\varepsilon_7} \right) + \sup_t \|h\|_{L_3(\Omega)}^2 \right) (a^2 (\|\theta_0\|_{L_\infty(\Omega^t)}) \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + \|v_0\|_{L_2(\Omega)}^2) \right), \\ t &\leq T. \end{aligned}$$

Proof. Multiplying (3.11)₁ by χ' , integrating over Ω , using boundary conditions (3.11)_{2,3},

(1.1)₅ and (1.1)₂, we obtain

$$\begin{aligned}
(3.13) \quad & \frac{1}{2} \frac{d}{dt} \|\chi'\|_{L_2(\Omega)}^2 + \nu \|\nabla \chi'\|_{L_2(\Omega)}^2 = \int_{\Omega} h_3 \chi'^2 dx \\
& - \int_{\Omega} (h_2 w_{,x_1} - h_1 w_{,x_2}) \chi' dx + \int_{\Omega} \alpha_{\theta} (\theta_{,x_1} f_2 - \theta_{,x_2} f_1) \chi' dx \\
& + \int_{\Omega} \alpha F \chi' dx - \int_{\Omega} v \cdot \nabla \tilde{\chi} \chi' dx + \int_{\Omega} h_3 \tilde{\chi} \chi' dx.
\end{aligned}$$

Now we estimate the terms on the r.h.s. of the above equality. Let $x' = (x_1, x_2)$. The first term we estimate by

$$\left| \int_{\Omega} h_3 \chi'^2 dx \right| \leq \varepsilon_1 \|\chi'\|_{L_6(\Omega)}^2 + \frac{c}{\varepsilon_1} \|\chi'\|_{L_2(\Omega)}^2 \|h_3\|_{L_3(\Omega)}^2,$$

the second by

$$\varepsilon_2 \|\chi'\|_{L_6(\Omega)}^2 + \frac{c}{\varepsilon_2} \|h\|_{L_3(\Omega)}^2 \|w_{,x'}\|_{L_2(\Omega)}^2,$$

the third by

$$\varepsilon_3 \|\chi'\|_{L_6(\Omega)}^2 + \frac{c}{\varepsilon_3} c_1^2 \|\theta_{,x}\|_{L_2(\Omega)}^2 \|f\|_{L_3(\Omega)}^2,$$

where we used (1.1'). The fourth by

$$\varepsilon_4 \|\chi'\|_{L_6(\Omega)}^2 + \frac{c}{\varepsilon_4} c_1^2 \|F\|_{L_{6/5}(\Omega)}^2,$$

where we used also (1.1').

To estimate the fifth term on the r.h.s. of (3.13) we integrate it by parts and use (1.1)_{2,5}.

Then it takes the form

$$I \equiv \int_{\Omega} v \cdot \nabla \chi' \tilde{\chi} dx.$$

Hence

$$|I| \leq \varepsilon_5 \|\nabla \chi'\|_{L_2(\Omega)}^2 + \frac{c}{\varepsilon_5} \|v\|_{L_6(\Omega)}^2 \|\tilde{\chi}\|_{L_3(\Omega)}^2.$$

Finally, the last term on the r.h.s. of (3.13) is bounded by

$$\varepsilon_6 \|\chi'\|_{L_6(\Omega)}^2 + \frac{c}{\varepsilon_6} \|h\|_{L_3(\Omega)}^2 \|\tilde{\chi}\|_{L_2(\Omega)}^2.$$

Using the above estimates in (3.13), assuming that $\varepsilon_1, \dots, \varepsilon_6$ are sufficiently small, integrating the result with respect to time and using (2.7)–(2.8) we obtain

$$\begin{aligned}
(3.14) \quad & \|\chi'\|_{V_2^0(\Omega_t)}^2 \leq c(\sup_t \|h\|_{L_3(\Omega)}^2 \|\chi'\|_{L_2(0,t;L_2(\Omega))}^2 \\
& + c_0^2 \sup_t \|h\|_{L_3(\Omega)}^2 + c_1^2 c_0^2 \sup_t \|f\|_{L_3(\Omega)}^2 + c_1^2 \|F\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\
& + c_0^2 \|\tilde{\chi}\|_{L_{\infty}(0,t;L_3(\Omega))}^2 + \sup_t \|h\|_{L_3(\Omega)}^2 \|\tilde{\chi}\|_{L_2(0,t;L_2(\Omega))}^2 + \|\chi(0)\|_{L_2(\Omega)}^2).
\end{aligned}$$

In view of (2.7) we have $\|\chi\|_{L_2(\Omega^t)} \leq cc_0$.

Using (3.10) and this fact we obtain from (3.14) the inequality

$$\begin{aligned}
(3.15) \quad & \|\chi\|_{V_2^0(\Omega^t)}^2 \leq c(c_0^2 \sup_t \|h\|_{L_3(\Omega)}^2 + c_1^2 c_0^2 \sup_t \|f\|_{L_3(\Omega)}^2 \\
& + c_1^2 \|F\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + c_0^2 \|\tilde{\chi}\|_{L_\infty(0,t;L_3(\Omega))}^2 + \sup_t \|h\|_{L_3(\Omega)}^2 \|\tilde{\chi}\|_{L_2(\Omega^t)}^2 \\
& + \|\tilde{\chi}\|_{V_2^0(\Omega^t)}^2 + \|\chi(0)\|_{L_2(\Omega)}^2).
\end{aligned}$$

Since $\tilde{\chi}$ is a solution of (3.9) and χ_* is described by (2.12)₂ we have the estimates by Lemma 2.8,

$$\begin{aligned}
(3.16) \quad & \int_0^t \|\tilde{\chi}(t')\|_{L_2(\Omega)}^2 dt' \leq c \int_0^t \|v'(t')\|_{L_2(S)}^2 dt' \leq c \int_0^t \|v'(t')\|_{H^1(\Omega)}^2 dt' \\
& \leq c(a^2(\|\theta_0\|_{L_\infty(\Omega)})\|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + \|v_0\|_{L_2(\Omega)}^2), \\
& \|\tilde{\chi}\|_{L_\infty(0,t;L_3(\Omega))} \leq c\|v'\|_{L_\infty(0,t;L_3(S))} \leq \varepsilon_7 \|v'\|_{L_\infty(0,t;H^1(\Omega))} + c\left(\frac{1}{\varepsilon_7}\right) \|v'\|_{L_\infty(0,t;L_2(\Omega))}, \\
& \|\tilde{\chi}\|_{V_2^0(\Omega^t)}^2 \leq c\left(\|\tilde{\chi}\|_{L_\infty(0,t;L_2(\Omega))}^2 + \int_0^t \|\tilde{\chi}(t')\|_{H^1(\Omega)}^2 dt'\right) \\
& \leq c\|v'\|_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))}^2 + c\|v'\|_{W_2^{1,1/2}(\Omega^t)}^2, \quad \varepsilon > 0.
\end{aligned}$$

Employing (3.16) in (3.15) yields (3.12). This concludes the proof. \square

Let us consider the problem

$$\begin{aligned}
(3.17) \quad & v_{2,x_1} - v_{1,x_2} = \chi \quad \text{in } \Omega', \\
& v_{1,x_1} + v_{2,x_2} = -h_3 \quad \text{in } \Omega', \\
& v' \cdot \bar{n}' = 0 \quad \text{on } S',
\end{aligned}$$

where $\Omega' = \Omega \cap \{\text{plane } x_3 = \text{const} \in (-a, a)\}$, $S' = S \cap \{\text{plane } x_3 = \text{const} \in (-a, a)\}$, x_3, t are treated as parameters, $\bar{n}' = (n_1, n_2)$.

Lemma 3.3. *Let the assumptions of Lemmas 2.3, 3.1, 3.2 be satisfied. Assume that (v, p, θ) is a weak solution to problem (1.1). Assume that*

$$\begin{aligned}
(3.18) \quad & c_1 \|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1 c_0 \|f\|_{L_\infty(0,t;L_3(\Omega))} + c_1 \|F\|_{L_2(0,t;L_{6/5}(\Omega))} \\
& + c_1 \|f_3\|_{L_2(0,t;L_{\frac{4}{3}}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)} \\
& + c_0^2 (c_1 \|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \leq k_1 < \infty, \\
& \|f\|_{L_2(0,t;L_3(\Omega))} \leq k_2 < \infty,
\end{aligned}$$

$t \leq T$. Then the following inequality

$$(3.19) \quad \|v'\|_{V_2^1(\Omega^t)}^2 \leq c[e^{cc_1^2 k_2^2}(c_0^2 \|h\|_{L_\infty(0,t;L_3(\Omega))}^2 + \psi(c_0)k_1^2) + \|v'\|_{L_2(\Omega;H^{1/2}(0,t))}^2]$$

holds, where $v' = (v_1, v_2)$, $t \leq T$ and ψ is an increasing positive function.

Proof. Assuming that ε_7 is sufficiently small in view of (3.1), (3.12) and Lemma 2.6 we obtain for solutions to problem (3.17) the inequality (see [9])

$$(3.20) \quad \begin{aligned} \|v'\|_{L_{10}(\Omega^T)}^2 &\leq c\|v'\|_{V_2^1(\Omega^t)}^2 \leq c[e^{cc_1^2 k_2^2}(c_0^2 \|h\|_{L_\infty(0,t;L_3(\Omega))}^2 + \psi(c_0)k_1^2) \\ &\quad + \|v'\|_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))}^2 + \|v'\|_{W_2^{1,1/2}(\Omega^t)}^2], \end{aligned}$$

where ε is arbitrary small number.

By interpolation inequalities

$$\|v'\|_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} \leq \varepsilon_1 \|v'\|_{L_\infty(0,t;H^1(\Omega))} + c(1/\varepsilon_1) \|v'\|_{L_\infty(0,t;L_2(\Omega))},$$

and

$$\|v'\|_{W_2^{1,1/2}(\Omega^t)} = \|v'\|_{L_2(0,t;H^1(\Omega))} + \|v'\|_{L_2(\Omega;H^{1/2}(0,t))},$$

where

$$\|v'\|_{L_2(0,t;H^1(\Omega))} \leq ck_1.$$

Then we obtain (3.19) from (3.20) for sufficiently small ε_1 . This concludes the proof. \square

Let us consider problem (1.1)_{1,2,4,5,7} in the form

$$(3.21) \quad \begin{aligned} v_{,t} - \operatorname{div} \mathbf{T}(v, p) &= -v' \cdot \nabla' v - wh + \alpha(\theta)f && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ v \cdot \bar{n} = 0, \quad \bar{n} \cdot \mathbf{D}(v) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2, && \text{on } S^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega. \end{aligned}$$

where $v' \cdot \nabla' = v_1 \partial_{x_1} + v_2 \partial_{x_2}$.

Lemma 3.4. Assume that (v, p, θ) is a weak solution to problem (1.1). Let the assumptions of Lemma 3.3 be satisfied. Let

$$\begin{aligned} \|f\|_{L_2(\Omega^t)} + \|v_0\|_{H^1(\Omega)} &\leq k_3 < \infty \\ H(t) = \|h\|_{L_\infty(0,t;L_3(\Omega))} + \|h\|_{L_{\frac{10}{3}}(\Omega^t)} &< \infty, \end{aligned}$$

$t \leq T$. Then there exists a constant $c_2 = c_2(c_0, c_1)$ such that for solutions to problem (3.21) the inequality

$$(3.22) \quad \|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \leq c_2 e^{cc_1^2 k_2^2} (H + 1 + k_1 + k_3)^2 + ck_3, \quad t \leq T,$$

holds.

Proof is the same as the proof of Lemma 3.3 in [7].

Finally, we obtain an estimate for h .

Lemma 3.5. *Let the assumptions of Lemma 3.4 be satisfied. Let*

$$(3.23) \quad \begin{aligned} c_1 \|f\|_{L_\infty(\Omega^t)} e^{cc_1^2 k_2^2} k_1 + c_1 \|g\|_{L_\sigma(\Omega^t)} + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega^t)} + \|h(0)\|_{W_\sigma^{2-1/\sigma}(\Omega)} &\leq k_4 < \infty, \\ c_1 \|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1 \|f_3\|_{L_2(0,t;L_{\frac{4}{3}}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} &\leq d < \infty, \\ c_1 \|f\|_{L_\varrho(\Omega^T)} + \|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} &\leq k_5 < \infty, \end{aligned}$$

for $t \leq T$. Then for d sufficiently small there exists a constant A such that

$$(3.24) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq A, \quad \frac{5}{3} < \sigma, \quad t \leq T,$$

$$(3.25) \quad \|\nabla p\|_{L_\varrho(\Omega^t)} + \|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \leq \varphi(A) + ck_5, \quad \frac{5}{3} \leq \varrho, \quad t \leq T,$$

where φ is some positive increasing function.

Proof. In view of Lemma 2.2 for solutions to problem (2.10) we have

$$(3.26) \quad \begin{aligned} &\|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq c(\|v \cdot \nabla h\|_{L_\sigma(\Omega^t)} \\ &\quad + \|h \cdot \nabla v\|_{L_\sigma(\Omega^t)} + \|\alpha_\theta \vartheta f\|_{L_\sigma(\Omega^t)} + \|\alpha g\|_{L_\sigma(\Omega^t)} \\ &\quad + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}). \end{aligned}$$

In view of the imbedding

$$(3.27) \quad \|v\|_{L_{10}(\Omega^t)} + \|\nabla v\|_{L_{\frac{10}{3}}(\Omega^t)} \leq c\|v\|_{W_2^{2,1}(\Omega^t)}.$$

and inequality (3.22) we estimate the first term on the r.h.s. of (3.26) by

$$\|v\|_{L_{10}(\Omega^t)} (\varepsilon_1 \|h\|_{W_\sigma^{2,1}(\Omega^t)} + c \left(\frac{1}{\varepsilon_1} \right) \|h\|_{L_2(\Omega^t)})$$

and the second by

$$\|\nabla v\|_{L_{\frac{10}{3}}(\Omega^t)} (\varepsilon_2 \|h\|_{W_\sigma^{2,1}(\Omega^t)} + c \left(\frac{1}{\varepsilon_2} \right) \|h\|_{L_2(\Omega^t)}).$$

In view of (2.6) and (1.1') the third and the fourth terms on the r.h.s. of (3.26) can be estimated by

$$cc_1 (\|f\|_{L_\infty(\Omega^t)} \|\vartheta\|_{L_\sigma(\Omega^t)} + \|g\|_{L_\sigma(\Omega^t)}) \equiv I.$$

We use (3.1) with notation (3.18). Then we obtain

$$I \leq cc_1 (\|f\|_{L_\infty(\Omega^t)} e^{cc_1^2 k_2^2} (k_1 + c_0 \|h\|_{L_\infty(0,t;L_3(\Omega))}) + \|g\|_{L_\sigma(\Omega^t)}).$$

We will use also the interpolation

$$\|h\|_{L_\infty(0,t;L_3(\Omega))} \leq \varepsilon_2 \|h\|_{W_\sigma^{2,1}(\Omega^t)} + c(1/\varepsilon_3) \|h\|_{L_2(\Omega^t)}.$$

Employing the above estimates in (3.26), assuming that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are sufficiently small and using (3.22) we obtain

$$(3.28) \quad \begin{aligned} & \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq \varphi(H) \|h\|_{L_2(\Omega^t)} \\ & + cc_1 (\|f\|_{L_\infty(\Omega^t)} e^{cc_1^2 k_2^2} k_1 + \|g\|_{L_\sigma(\Omega^t)}) + c \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}, \end{aligned}$$

where φ is an increasing positive function depending on H and on constants $c_0, c_1, k_1, \dots, k_5$.

Using notation (3.23)₁ we have

$$(3.29) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq \varphi(H) \|h\|_{L_2(\Omega^t)} + ck_4.$$

We want to estimate $\|h\|_{L_2(\Omega^t)}$ by applying (3.2). For this purpose we need to estimate $\|\nabla \theta\|_{L_2(0,t;L_3(\Omega))}$. Hence we consider problem (1.1)_{3,6,7} and we are looking for solutions of this problem such that $\theta \in W_\varrho^{2,1}(\Omega^t)$ with so large ϱ that

$$(3.30) \quad \|\nabla \theta\|_{L_2(0,t;L_3(\Omega))} \leq c \|\theta\|_{W_\varrho^{2,1}(\Omega^t)}.$$

We see that (3.30) holds for $\varrho \geq \frac{5}{3}$.

Considering problem (1.1)_{3,6,7} we have

$$(3.31) \quad \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \leq c \|v \cdot \nabla \theta\|_{L_\varrho(\Omega^t)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)}.$$

The first term on the r.h.s. we estimate by

$$\|v\|_{L_{\varrho\lambda_1}(\Omega^t)} \|\nabla \theta\|_{L_{\varrho\lambda_2}(\Omega^t)} \equiv I_1,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\varrho\lambda_1 = 10$.

Using the interpolation inequality

$$\|\nabla \theta\|_{L_{\varrho\lambda_2}(\Omega^t)} \leq \varepsilon_4 \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} + c \left(\frac{1}{\varepsilon_4} \right) \|\theta\|_{L_2(\Omega^t)}$$

which holds for $\frac{5}{\varrho} - \frac{5}{\varrho\lambda_2} < 1$ so for $\frac{5}{\varrho\lambda_1} < 1$. Hence

$$I_1 \leq \|v\|_{L_{10}(\Omega^t)} (\varepsilon_4 \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} + c \left(\frac{1}{\varepsilon_4} \right) \|\theta\|_{L_2(\Omega^t)}).$$

Using the estimate in (3.31), assuming that ε_4 is sufficiently small, using (3.27) and (3.22), we obtain

$$(3.32) \quad \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \leq \varphi(H) + c\|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega^T)}.$$

Similarly by [4, Theorem 2.2]

$$(3.33) \quad \|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\nabla p\|_{L_\varrho(\Omega^t)} \leq \varphi(H) + c_1\|f\|_{L_\varrho(\Omega^T)} + c\|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)}$$

Let us consider (3.29). In view of (3.2) we estimate the norm $\|h\|_{L_2(\Omega^t)}$, where

$$\|\nabla v\|_{L_2(0,t;L_3(\Omega))} + \|\nabla \theta\|_{L_2(0,t;L_3(\Omega))} \leq \varphi(H) + ck_5$$

Then (3.29) takes the form

$$(3.34) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq \varphi(H)d + ck_4,$$

where φ is an increasing positive function.

Let σ be such that

$$H = \|h\|_{L_\infty(0,t;L_3(\Omega))} + \|h\|_{L_{\frac{10}{3}}(\Omega^t)} \leq c\|h\|_{W_\sigma^{2,1}(\Omega^t)},$$

which holds for $\sigma > \frac{5}{3}$.

Then (3.34) takes the form

$$(3.35) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq \varphi(\|h\|_{W_\sigma^{2,1}(\Omega^t)})d + ck_4$$

Hence for d sufficiently small there exists a constant A such that

$$(3.36) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq A, \quad t \leq T.$$

By (3.36), (3.32) and (3.33) the proof is complete. \square

Proof of the main Theorem

Now we want to increase regularity described by (3.25). Assume $10 \leq \varrho < \infty$. In view of [5, Theorem 2.1] for a solution v to problem (1.1) we have

$$(3.37) \quad \begin{aligned} & \|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\nabla p\|_{L_\varrho(\Omega^t)} \leq c(\|v \cdot \nabla v\|_{L_\varrho(\Omega^t)} \\ & \quad + \|\alpha(\theta)f\|_{L_\varrho(\Omega^t)} + \|v_0\|_{W_\varrho^{2-2/\varrho}(\Omega)}). \end{aligned}$$

We estimate the first term on the r.h.s. of (3.37) by

$$(3.38) \quad \begin{aligned} & \|v\|_{L_\infty(\Omega^t)} \|\nabla v\|_{L_\varrho(\Omega^t)} \\ & \leq c\|v\|_{W_5^{2,1}(\Omega^t)} (\varepsilon_1\|v\|_{W_\varrho^{2,1}(\Omega^t)} + c(1/\varepsilon_1)\|v\|_{L_2(\Omega^t)}) \end{aligned}$$

and the second by

$$(3.39) \quad c_1 \|f\|_{L_\infty(\Omega^t)}.$$

Assuming that ε_1 is sufficiently small and using (3.37)–(3.39) we obtain

$$(3.40) \quad \|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\nabla p\|_{L_\varrho(\Omega^t)} \leq B_1,$$

where B_1 is a constant depending on constants from imbedding theorems and data. Similarly by [3, Ch. 4, Sect. 9, Th. 9.1] we obtain

$$(3.41) \quad \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \leq B_2.$$

Now we want to increase regularity described by (3.24).

There exist $p' > \sigma$, $p'' > \frac{5}{2}$ such that

$$\frac{5}{\varrho} - \frac{5}{p'} < 1, \quad \frac{5}{\varrho} - \frac{5}{p''} < 1.$$

Hence $p = \max\{p', p''\}$ satisfies

$$(3.42) \quad p > \sigma, \quad p > \frac{5}{2}, \quad \frac{5}{\varrho} - \frac{5}{p} < 1.$$

Similarly we can prove that there exists q such that

$$(3.43) \quad q > \sigma, \quad q > 5 \quad \text{and} \quad \frac{5}{\varrho} - \frac{5}{q} < 2.$$

Define \bar{p}, \bar{q} such that $\frac{1}{\bar{p}} + \frac{1}{\bar{p}} = \frac{1}{\sigma}$, $\frac{1}{\bar{q}} + \frac{1}{\bar{q}} = \frac{1}{\sigma}$. Assume $\frac{5}{3} < \sigma < \infty$. In view of Theorem 2.1 for a solution to problem (2.10) we have

$$(3.44) \quad \begin{aligned} & \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leq c(\|v \cdot \nabla h\|_{L_\sigma(\Omega^t)} \\ & + \|h \cdot \nabla v\|_{L_\sigma(\Omega^t)} + \|\alpha_\theta \vartheta f\|_{L_\sigma(\Omega^t)} + \|\alpha g\|_{L_\sigma(\Omega^t)} \\ & + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}). \end{aligned}$$

By (3.42) and (3.43) we estimate the first term on the r.h.s. of (3.44) by

$$(3.45) \quad \begin{aligned} & \|v\|_{L_q(\Omega^t)} \|\nabla h\|_{L_{\bar{q}}(\Omega^t)} \leq c\|v\|_{W_\varrho^{2,1}(\Omega^t)} (\varepsilon_2 \|h\|_{W_\sigma^{2,1}(\Omega^t)} \\ & + c(\varepsilon_2) \|h\|_{L_2(\Omega^t)}) \end{aligned}$$

the second by

$$(3.46) \quad \|\nabla v\|_{L_p(\Omega^t)} \|h\|_{L_{\bar{p}}(\Omega^t)} \leq c\|v\|_{W_\varrho^{2,1}(\Omega^t)} (\varepsilon_3 \|h\|_{W_\sigma^{2,1}(\Omega^t)} + c(\varepsilon_3) \|h\|_{L_2(\Omega^t)})$$

the third by

$$(3.47) \quad c_1 \|f\|_{L_\infty(\Omega^t)} (\varepsilon_4 \|\vartheta\|_{W_\sigma^{2,1}(\Omega^t)} + c(\varepsilon_4) \|\vartheta\|_{L_2(\Omega^t)})$$

the fourth by

$$(3.48) \quad c_1 \|g\|_{L_\sigma(\Omega^t)}.$$

In view of [3, Ch. 4, Sect. 9, Th. 9.1] for any solution to problem (2.13) we have

$$(3.49) \quad \begin{aligned} \|\vartheta\|_{W_\sigma^{2,1}(\Omega^t)} &\leq c(\|v \cdot \nabla \vartheta\|_{L_\sigma(\Omega^t)} \\ &\quad + \|h \cdot \nabla \theta\|_{L_\sigma(\Omega^t)} + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega^t)}). \end{aligned}$$

By (3.42) and (3.43) we estimate the first term on the r.h.s. of (3.49) by

$$(3.50) \quad c \|v\|_{W_\rho^{2,1}(\Omega^t)} (\varepsilon_5 \|\vartheta\|_{W_\sigma^{2,1}(\Omega^t)} + c(\varepsilon_5) \|\vartheta\|_{L_2(\Omega^t)})$$

the second by

$$(3.51) \quad c \|\theta\|_{W_\rho^{2,1}(\Omega^t)} (\varepsilon_6 \|h\|_{W_\sigma^{2,1}(\Omega^t)} + c(\varepsilon_6) \|h\|_{L_2(\Omega^t)}).$$

We choose r such that $\frac{5}{3} < r < \frac{10}{3}$ and $r \leq \sigma$. By (3.1), the imbedding

$$\|h\|_{L_\infty(0,t;L_3(\Omega))} \leq c \|h\|_{W_r^{2,1}(\Omega^t)}$$

and (3.24) there exists a constant B_3 depending on constants from imbedding theorems and data such that

$$(3.52) \quad \|h\|_{L_2(\Omega^t)} + \|\vartheta\|_{L_2(\Omega^t)} \leq B_3.$$

Assuming that $\varepsilon_2 - \varepsilon_6$ are sufficiently small and using (3.44)–(3.52) we obtain

$$(3.53) \quad \|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} + \|\vartheta\|_{W_\sigma^{2,1}(\Omega^t)} \leq B_4,$$

where B_4 is some constant depending on data. By (3.40), (3.41) and (3.53) the proof is finished. \square

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